

Supplementary Information – Quantum Advantage in Postselected Metrology

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SUPPLEMENTARY INFORMATION

Supplementary Note 1 – Expressing the postselected quantum Fisher information in terms of the KD distribution

As shown in the Results section of our main paper, the postselected quantum Fisher information is given by

$$\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) = 4 \langle \dot{\psi}_\theta^{\text{ps}} | \dot{\psi}_\theta^{\text{ps}} \rangle \frac{1}{p_\theta^{\text{ps}}} - 4 \left| \langle \dot{\psi}_\theta^{\text{ps}} | \dot{\psi}_\theta^{\text{ps}} \rangle \right|^2 \frac{1}{(p_\theta^{\text{ps}})^2}, \quad (1)$$

where nonrenormalized postselected quantum state is $|\psi_\theta^{\text{ps}}\rangle = \hat{F}\hat{U}(\theta)|\Psi_0\rangle$, where $|\Psi_0\rangle\langle\Psi_0| \equiv \hat{\rho}_0$. $p_\theta^{\text{ps}} = \text{Tr}(\hat{F}\hat{\rho}_\theta)$ is the probability of postselection.

In this supplementary note, we show that Supplementary Equation 1 can be expressed in terms of the doubly-extended KD distribution:

$$\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) = 4 \sum_{\substack{a,a', \\ f \in \mathcal{F}^{\text{ps}}}} \frac{q_{a,a',f}^{\hat{\rho}_\theta}}{p_\theta^{\text{ps}}} aa' - 4 \left| \sum_{\substack{a,a', \\ f \in \mathcal{F}^{\text{ps}}}} \frac{q_{a,a',f}^{\hat{\rho}_\theta}}{p_\theta^{\text{ps}}} a \right|^2, \quad (2)$$

The first term of the quantum Fisher information (Supplementary Equation 1) is

$$\frac{4}{p_\theta^{\text{ps}}} \langle \dot{\psi}_\theta^{\text{ps}} | \dot{\psi}_\theta^{\text{ps}} \rangle = \frac{4}{p_\theta^{\text{ps}}} \text{Tr}(\hat{F}\dot{\hat{U}}(\theta)\hat{\rho}_0\dot{\hat{U}}^\dagger(\theta)\hat{F}^\dagger) = \frac{4}{p_\theta^{\text{ps}}} \text{Tr}(\hat{F}\hat{A}\hat{\rho}_\theta\hat{A}) \quad (3)$$

$$= \frac{4}{p_\theta^{\text{ps}}} \text{Tr}\left(\sum_a |a\rangle\langle a| a \hat{\rho}_\theta \sum_{a'} |a'\rangle\langle a'| a' \sum_{f \in \mathcal{F}^{\text{ps}}} |f\rangle\langle f|\right), \quad (4)$$

where, in Supplementary Equation 4, we have expressed \hat{A} and \hat{F} in their corresponding eigendecompositions. This expression can be rewritten in terms of the doubly extended Kirkwood-Dirac quasiprobability distribution ($q_{a,a',f}^{\hat{\rho}} = \langle f|a\rangle\langle a|\hat{\rho}|a'\rangle\langle a'|f\rangle$):

$$\frac{4}{p_\theta^{\text{ps}}} \sum_{\substack{a,a', \\ f \in \mathcal{F}^{\text{ps}}}} \text{Tr}\left(aa' q_{a,a',f}^{\hat{\rho}_\theta} \frac{|a\rangle\langle f|}{\langle f|a\rangle}\right) = \frac{4}{p_\theta^{\text{ps}}} \sum_{\substack{a,a', \\ f \in \mathcal{F}^{\text{ps}}}} q_{a,a',f}^{\hat{\rho}_\theta} aa'. \quad (5)$$

Similarly, the second term of Supplementary Equation 1 is

$$\frac{4}{(p_\theta^{\text{ps}})^2} \left| \langle \dot{\psi}_\theta^{\text{ps}} | \dot{\psi}_\theta^{\text{ps}} \rangle \right|^2 = \frac{4}{(p_\theta^{\text{ps}})^2} \left| \text{Tr}(\hat{F}\hat{\rho}_\theta\hat{A}) \right|^2 = \frac{4}{(p_\theta^{\text{ps}})^2} \left| \sum_{\substack{a,a', \\ f \in \mathcal{F}^{\text{ps}}}} q_{a,a',f}^{\hat{\rho}_\theta} a \right|^2. \quad (6)$$

Combining the expressions above gives Supplementary Equation 2:

$$\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) = 4 \sum_{\substack{a,a', \\ f \in \mathcal{F}^{\text{ps}}}} \frac{q_{a,a',f}^{\hat{\rho}_\theta}}{p_\theta^{\text{ps}}} aa' - 4 \left| \sum_{\substack{a,a', \\ f \in \mathcal{F}^{\text{ps}}}} \frac{q_{a,a',f}^{\hat{\rho}_\theta}}{p_\theta^{\text{ps}}} a \right|^2. \quad (7)$$

Supplementary Note 2 – Proof of Theorem 2

Here, we prove Theorem 2. First, we upper-bound the right-hand side of Supplementary Equation 2, assuming that all $q_{a,a',f}^{\hat{\rho}_\theta}/p_\theta^{\text{ps}} \in [0, 1]$. We label the M eigenvalues of \hat{A} and arrange them in increasing order: a_1, a_2, \dots, a_M , such that $a_1 \equiv a_{\min}$ and $a_M \equiv a_{\max}$. Initially, we assume that the 0-point of the eigenvalue axis is set such that $a_1 = 0$ and $a_M = \Delta a$. In this scenario, all the components of the first term of Supplementary Equation 2 are nonnegative. We temporarily ignore the form of $q_{a,a',f}^{\hat{\rho}_\theta}/p_\theta^{\text{ps}}$, and treat this ratio as a general quasiprobability distribution. Then, $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})$ maximizes when $q_{a,a',f}^{\hat{\rho}_\theta}/p_\theta^{\text{ps}}$ vanishes at all a' values except $a' = a_{\max}$. We define $q_a \equiv \sum_{a', f \in \mathcal{F}^{\text{ps}}} q_{a,a',f}^{\hat{\rho}_\theta}/p_\theta^{\text{ps}}$,

such that all $q_a \in [0, 1]$ and $\sum_a q_a = 1$. If $q_{a,a',f}^{\hat{\rho}_\theta}/p_\theta^{\text{ps}}$ is nonzero only when $a' = a_{\text{max}}$, Supplementary Equation 2 becomes

$$\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) = 4a_M \sum_a q_a a - 4 \left(\sum_a q_a a \right)^2. \quad (8)$$

Expanding each sum, we obtain

$$\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) = 4a_M(q_{a_1}a_1 + K + q_{a_M}a_M) - 4(q_{a_1}a_1 + K + q_{a_M}a_M)^2 \quad (9)$$

$$= 4a_M(K + q_{a_M}a_M) - 4(K + q_{a_M}a_M)^2, \quad (10)$$

where we used $q_{a_1}a_1 = 0$ and defined $K \equiv \sum_{a \in \{a_2, \dots, a_{M-1}\}} q_a a \leq a_M$. As \hat{A} is not totally degenerate, $a_M \neq 0$, and Supplementary Equation 10 is maximized when $q_{a_M} = (a_M - 2K)/(2a_M)$. This yields

$$\max\{\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})\} = a_M^2 = (\Delta a)^2, \quad (11)$$

where we have recalled that $a_M = \Delta a$.

We are left with proving that we can always set $a_1 = 0$ and $a_M = \Delta a$. We continue to assume that $q_{a,a',f}^{\hat{\rho}_\theta}/p_\theta^{\text{ps}} \in [0, 1]$, and we shift all the eigenvalues by a constant real value δ_a . The effect on $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})$ is

$$\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) \rightarrow 4 \sum_{\substack{a,a', \\ f \in \mathcal{F}^{\text{ps}}}} \frac{q_{a,a',f}^{\hat{\rho}_\theta}}{p_\theta^{\text{ps}}} (a + \delta_a)(a' + \delta_a) - 4 \left[\sum_{\substack{a,a', \\ f \in \mathcal{F}^{\text{ps}}}} \frac{q_{a,a',f}^{\hat{\rho}_\theta}}{p_\theta^{\text{ps}}} (a + \delta_a) \right]^2 \quad (12)$$

$$= 4 \sum_{\substack{a,a', \\ f \in \mathcal{F}^{\text{ps}}}} \frac{q_{a,a',f}^{\hat{\rho}_\theta}}{p_\theta^{\text{ps}}} a a' - 4 \left[\sum_{\substack{a,a', \\ f \in \mathcal{F}^{\text{ps}}}} \frac{q_{a,a',f}^{\hat{\rho}_\theta}}{p_\theta^{\text{ps}}} a \right]^2 + 4\delta_a \left(\sum_{\substack{a,a', \\ f \in \mathcal{F}^{\text{ps}}}} \frac{q_{a,a',f}^{\hat{\rho}_\theta}}{p_\theta^{\text{ps}}} a - \sum_{\substack{a,a', \\ f \in \mathcal{F}^{\text{ps}}}} \frac{q_{a,a',f}^{\hat{\rho}_\theta}}{p_\theta^{\text{ps}}} a' \right) = \mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}). \quad (13)$$

The last equality holds because $q_{a,a',f}^{\hat{\rho}_\theta} = (q_{a',a,f}^{\hat{\rho}_\theta})^*$ generally and we are assuming that $q_{a,a',f}^{\hat{\rho}_\theta} \in \mathbb{R}$. Consequently, if all $q_{a,a',f}^{\hat{\rho}_\theta}/p_\theta^{\text{ps}} \in [0, 1]$, then $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) \leq (\Delta a)^2$. The second term of Supplementary Equation 2 cannot be decreased by imaginary values in $q_{a,a',f}^{\hat{\rho}_\theta}$. Moreover, the first term is necessarily real and nonnegative. Thus imaginary elements $q_{a,a',f}^{\hat{\rho}_\theta}$ cannot increase $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})$. If $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) > (\Delta a)^2$, then $q_{a,a',f}^{\hat{\rho}_\theta}$ must have negative entries.

Supplementary Note 3 – Infinite postselected quantum Fisher information

Here, we show that the postselected quantum Fisher information $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})$ can approach infinity. The proof is by example; other examples might exist.

We assume that the generator \hat{A} has $M \geq 3$ eigenvalues that are not all identical. We also assume that we possess an estimate θ_0 that lies close to the true value of θ : $\delta_\theta \equiv \theta - \theta_0$, with $|\delta_\theta| \ll 1$. (The derivation of the quantum Fisher information also rests on the assumption that one has access to such an estimate [1].)

By Supplementary Equations 1, 3 and 6,

$$\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) = \frac{4}{p_\theta^{\text{ps}}} \text{Tr} \left(\hat{F} \hat{A} \hat{U}(\theta) \hat{\rho}_0 \hat{U}(\theta)^\dagger \hat{A} \right) - \frac{4}{(p_\theta^{\text{ps}})^2} \left| \text{Tr} \left(\hat{F} \hat{U}(\theta) \hat{\rho}_0 \hat{U}(\theta)^\dagger \hat{A} \right) \right|^2. \quad (14)$$

We now choose \hat{F} and $\hat{\rho}_0$ such that $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})$ approaches infinity. Crudely, p_θ^{ps} must approach 0 while $\text{Tr}(\hat{F} \hat{A} \hat{U}(\theta) \hat{\rho}_0 \hat{U}(\theta)^\dagger \hat{A})$ either stays constant or approaches 0 more slowly. We label the M eigenvalues of \hat{A} and arrange them in increasing order: a_1, a_2, \dots, a_M , such that $a_1 \equiv a_{\text{min}}$ and $a_M \equiv a_{\text{max}}$.

First, we choose $\hat{F} = |f_1\rangle \langle f_1| + |f_2\rangle \langle f_2|$, where

$$|f_1\rangle \equiv \frac{|a_{\text{max}}\rangle + |a_{\text{min}}\rangle}{\sqrt{2}}, \quad (15)$$

$$|f_2\rangle \equiv \frac{\frac{i}{\sqrt{2}}(|a_{\text{max}}\rangle - |a_{\text{min}}\rangle) + |a_k\rangle}{\sqrt{2}}, \quad (16)$$

and $|a_k\rangle \neq |a_{\max}\rangle, |a_{\min}\rangle$. We also choose $\hat{\rho}_0 = |\Psi_0\rangle\langle\Psi_0|$ such that

$$|\Psi_0\rangle \equiv |\Psi_0(\theta_0, \phi)\rangle = \hat{U}^\dagger(\theta_0) \frac{1}{\sqrt{2}} \left\{ [\cos(\phi) - \sin(\phi)] \frac{i}{\sqrt{2}} (|a_{\min}\rangle - |a_{\max}\rangle) + [\cos(\phi) + \sin(\phi)] |a_k\rangle \right\}. \quad (17)$$

$\phi \approx 0$ is a parameter that can be tuned to maximize the postselected Fisher information for a given approximation accuracy δ_θ . As ϕ is a parameter of the input state, variations in the Fisher information with ϕ will reflect the effects of disturbances to the input state. Substituting the expressions for \hat{F} and $\hat{\rho}_0$ into Supplementary Equation 14, we find

$$\begin{aligned} \mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) = & 8 \left\{ 5 - 2 \cos(2\phi) \left(\cos[(a_M - a_k)\delta_\theta] + \cos[(a_k - a_1)\delta_\theta] \right) + \cos[(a_M - a_1)\delta_\theta] [\sin(2\phi) - 1] - \sin(2\phi) \right\}^{-2} \\ & \times \left\{ 2a_M^2 - a_M a_k + a_k^2 + 2a_1^2 - (3a_M + a_k)a_1 + (a_M - a_k)(a_k - a_1) \cos(4\phi) \left(\cos[(a_M - a_1)\delta_\theta] - 1 \right) \right. \\ & + (a_M - a_k)(a_k - a_1) \cos[(a_M - a_1)\delta_\theta] + 2(a_M - a_1) \cos(2\phi) \left((a_1 - a_k) \cos[(a_M - a_k)\delta_\theta] \right. \\ & + (a_k - a_M) \cos[(a_k - a_1)\delta_\theta] \left. \right) - 2(a_M - a_1)^2 \sin(2\phi) + (a_M - a_1) \left((a_k - a_1) \cos[(a_M - a_k)\delta_\theta] \right. \\ & \left. \left. + (a_M - a_k) \cos[(a_k - a_1)\delta_\theta] \right) \sin(4\phi) \right\}. \end{aligned} \quad (18)$$

The postselection probability is

$$p_\theta^{\text{ps}} = \frac{1}{8} \left\{ 5 - 2 \cos(2\phi) \left(\cos[(a_M - a_k)\delta_\theta] + \cos[(a_k - a_1)\delta_\theta] \right) + \cos[(a_M - a_1)\delta_\theta] [\sin(2\phi) - 1] - \sin(2\phi) \right\}. \quad (19)$$

In the limit as our estimate θ_0 approaches the true value of θ , such that $\delta_\theta \rightarrow 0$,

$$\lim_{\delta_\theta \rightarrow 0} p_\theta^{\text{ps}} = \sin^2(\phi), \quad (20)$$

$$\lim_{\delta_\theta \rightarrow 0} \mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) = \frac{(\cot(\phi) - 1)^2}{2} (\Delta a)^2, \text{ and} \quad (21)$$

$$\lim_{\delta_\theta \rightarrow 0} p_\theta^{\text{ps}} \times \mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) = \frac{1}{2} [1 - \sin(2\phi)] (\Delta a)^2. \quad (22)$$

In the limit as $\phi \rightarrow 0$,

$$\lim_{\phi \rightarrow 0} \left[\lim_{\delta_\theta \rightarrow 0} p_\theta^{\text{ps}} \right] = 0, \quad (23)$$

$$\lim_{\phi \rightarrow 0} \left[\lim_{\delta_\theta \rightarrow 0} \mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) \right] = \infty, \text{ and} \quad (24)$$

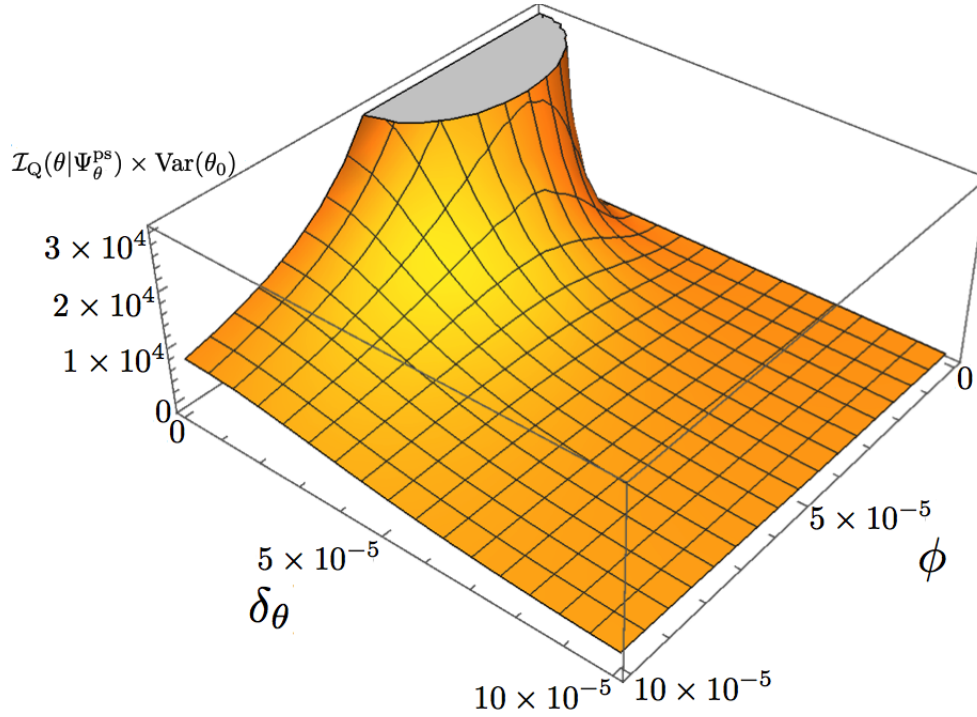
$$\lim_{\phi \rightarrow 0} \left[\lim_{\delta_\theta \rightarrow 0} p_\theta^{\text{ps}} \times \mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) \right] = \frac{1}{2} (\Delta a)^2. \quad (25)$$

According to Supplementary Equation 24, if first δ_θ and then ϕ approaches 0 in Supplementary Equation 18, $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})$ approaches infinity.

There are a few points to note. First, $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})$ diverges in the two ordered limits. In any real experiment, one could not blindly set $\phi = 0$, but would have to choose ϕ based on an estimate of θ . Second, if $\delta_\theta \approx 0$, then $\theta_0 \approx \theta$, and the pre-experiment variance of our initial estimate θ_0 , $\text{Var}(\theta_0)$, must be small. That is, we begin the experiment with much information about θ . Guided by the Cramér-Rao bound, we expect that, in a useful experiment, $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})$ would grow large, while $1/\text{Var}(\theta_0) < \mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})$. Supplementary Figure 1 shows $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) \times \text{Var}(\theta_0)$ as a function of ϕ and δ_θ for an experiment where $a_1 = -1$, $a_k = 1$, $a_M = 3$ and $\text{Var}(\theta_0) = 10^{-6}$. If θ_0 is within a few $\sigma_{\theta_0} \equiv \sqrt{\text{Var}(\theta_0)}$ of θ , then $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) \times \text{Var}(\theta_0) \gg 1$. Supplementary Figure 1 shows that large values of $1/\delta_\theta$ can result in even larger values of $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})$. Supplementary Figure 1 also illustrates the effect of input-state disturbances of ϕ on $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) \times \text{Var}(\theta_0)$. Third, while the theoretical strategy investigated in this appendix achieves an infinite postselected quantum Fisher information, the postselection also “wastes” information as $\lim_{\phi \rightarrow 0} [\lim_{\delta_\theta \rightarrow 0} p_\theta^{\text{ps}} \times \mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})] < (\Delta a)^2$. If \hat{A} possesses certain properties, it is possible to avoid wasting information through the postselection; we show how in the following appendix.

Supplementary Note 4 – Infinite postselected quantum Fisher information without loss of information

If the generator \hat{A} has $M \geq 4$ eigenvalues, and the minimum and maximum eigenvalues are both at least doubly degenerate, then $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})$ can approach infinity without information’s being lost in the events discarded by postselection. We show how below.



Supplementary Figure 1. **Scaled postselected quantum Fisher information.** The figure shows the postselected quantum Fisher information (Supplementary Equation 18) multiplied by the pre-experiment variance $\text{Var}(\theta_0)$ as a function of ϕ and δ_θ . For small values of δ_θ and ϕ , the value of $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) \times \text{Var}(\theta_0)$ diverges. The eigenvalues a_1 , a_k and a_M are set to -1 , 1 and 3 , respectively. $\text{Var}(\theta_0)$ was set to 1×10^{-6} .

First, we assign the orthonormal eigenvectors $|a_{\min_1}\rangle$ and $|a_{\min_2}\rangle$ to the eigenvalues $a_1 = a_{\min}$ and $a_2 = a_{\min}$, respectively. Here, we have reused the eigenvalue notation from Supp. Mat. . Similarly, we assign the orthonormal eigenvectors $|a_{\max_1}\rangle$ and $|a_{\max_2}\rangle$ to the eigenvalues $a_M = a_{\max}$ and $a_{M-1} = a_{\max}$, respectively. Second, we set $\hat{F} = |f_1\rangle\langle f_1| + |f_2\rangle\langle f_2|$, where

$$|f_1\rangle \equiv \frac{|a_{\max_2}\rangle - |a_{\min_1}\rangle}{\sqrt{2}}, \quad (26)$$

$$|f_2\rangle \equiv \frac{|a_{\min_2}\rangle - |a_{\max_1}\rangle}{\sqrt{2}}. \quad (27)$$

We also choose $|\Psi_0\rangle$ such that

$$|\Psi_0(\theta_0, \phi)\rangle = \hat{U}^\dagger(\theta_0) \frac{1}{2} \{ [\cos(\phi) - \sin(\phi)](|a_{\max_2}\rangle + |a_{\min_2}\rangle) + [\sin(\phi) + \cos(\phi)](|a_{\max_1}\rangle + |a_{\min_1}\rangle) \}. \quad (28)$$

As in App. , $\phi \approx 0$ is a parameter that can be tuned to maximize $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})$ for a given approximation accuracy of δ_θ .

Substituting the expressions for \hat{F} and $\hat{\rho}_0$ into Supplementary Equation 14, we find

$$\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) = \frac{\sin^2(2\phi)(a_M - a_1)^2}{(1 - \cos(2\phi) \cos[(a_M - a_1)\delta_\theta])^2}. \quad (29)$$

The postselection probability is

$$p_\theta^{\text{ps}} = \frac{1}{2} \left\{ 1 - \cos(2\phi) \cos[(a_M - a_1)\delta_\theta] \right\}. \quad (30)$$

Again, we investigate the limit as our estimate θ_0 approaches the true value of θ :

$$\lim_{\delta_\theta \rightarrow 0} p_\theta^{\text{ps}} = \sin^2(\phi), \quad (31)$$

$$\lim_{\delta_\theta \rightarrow 0} \mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) = \cot^2(\phi)(\Delta a)^2, \text{ and} \quad (32)$$

$$\lim_{\delta_\theta \rightarrow 0} p_\theta^{\text{ps}} \times \mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) = \cos^2(\phi)(\Delta a)^2. \quad (33)$$

In the limit as $\phi \rightarrow 0$,

$$\lim_{\phi \rightarrow 0} \left[\lim_{\delta_\theta \rightarrow 0} p_\theta^{\text{ps}} \right] = 0, \quad (34)$$

$$\lim_{\phi \rightarrow 0} \left[\lim_{\delta_\theta \rightarrow 0} \mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) \right] = \infty, \text{ and} \quad (35)$$

$$\lim_{\phi \rightarrow 0} \left[\lim_{\delta_\theta \rightarrow 0} p_\theta^{\text{ps}} \times \mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) \right] = (\Delta a)^2. \quad (36)$$

In conclusion, the above strategy allows us to obtain an infinite value for $\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})$, while $p_\theta^{\text{ps}} \times \mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) = (\Delta a)^2$. No information is lost in the postselection. As in Supplementary Note 3, the results hold for the two ordered limits.

$$\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}}) \times \text{Var}(\theta_0)$$

Supplementary references

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- [1] Samuel L. Braunstein and Carlton M. Caves, “Statistical distance and the geometry of quantum states,” Phys. Rev. Lett. **72**, 3439–3443 (1994).